Approximate Atkin-Serre Conjecture

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Abstract: Let $\lambda(m)$ be the $m$th coefficient of a modular form $f(z) = \sum_{m \geq 1} \lambda(m)q^m$ of weight $k \geq 4$, let $p^n$ be a prime power, and let $\varepsilon > 0$ be a small number. An approximate of the Atkin-Serre conjecture on the lower bound of the form $|\lambda(p^n)| \geq p^{(k-1)n/2 - 2k + 2\varepsilon}$ is presented in this note.

1 Introduction

The properties of the Fourier coefficients $\lambda : \mathbb{N} \rightarrow \mathbb{C}$ of modular forms $f(s) = \sum_{n \geq 1} \lambda(n)q^n = \lambda(1)q + \lambda(2)q^2 + \lambda(3)q^3 + \lambda(4)q^4 + \cdots$, (1)

where $s \in \mathbb{C}$ is a complex number in the upper half plane, $q = e^{2\pi is}$, are the topics of many studies. Basic information on modular forms, classified by various parameters such as level $N \geq 1$, weight $k \geq 1$, et cetera, and other data are archived in [6]. The corresponding $L$-function $L(s, f) = \sum_{n \geq 1} \lambda(n)n^{-s}$ is analytic on the complex half plane $\mathcal{H}_f = \{s \in \mathbb{C} : \Re(s) > (k + 1)/2\}$, and its functional equation

$$\xi(s) = (2\pi)^{-s}\Gamma(s)L(s, f), \quad \xi(s) = \xi(k - s) \quad (2)$$

facilitates an analytic continuation to the entire complex plane, see [5] p. 376]. One of the basic property of the Fourier coefficients is the dynamic range of its magnitude

$$-L \leq \lambda(p^n) \leq U, \quad (3)$$

where $L > 0$ and $U > 0$ are the lower bounds and upper bounds respectively. The earliest results for the upper bounds seems to be the Hecke estimate

$$|\lambda(p^n)| \leq cp^{(k-1)n}, \quad (4)$$

where $p^n$ is a prime power, $c > 0$ is a constant, and $k \geq 1$ is the weight, see [12 Theorem 4], [2 Proposition 5.4], et cetera. After many partial results by many authors, this line of research culminated with the effective upper bound (known as Deligne theorem)

$$|\lambda(p^n)| \leq 2p^{(k-1)n/2 + \varepsilon}, \quad (5)$$

where $\varepsilon > 0$ is a small number. Furthermore, the partial upper bound

$$|\lambda(p^n)| \leq 2p^{(k-1)n/2} \left(\log p^n\right)^{-1/2 + o(1)}, \quad (6)$$

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on a subset of primes of density 1, was proved in [7, Theorem 1.1].

On the other hand, the lower bounds for nonvanishing Fourier coefficients have no effective results. However, the Atkin-Serre conjecture claims the lower bound

\[ |\lambda(p^n)| \geq p^{(k-3)n/2-\varepsilon}, \]  

(7)

where \( k \geq 4 \), and \( \varepsilon > 0 \) is an arbitrary small number, see [10, [4, Equation 1.3], et alii. A much weaker, almost trivial, but unconditional lower bound proved in [8, Theorem 1], has the form

\[ |\lambda(m)| \geq (\log m)^5, \]  

(8)

where \( m \) is an integer for which \( \lambda(m) \) is odd, and \( c > 0 \) is an effectively computable absolute constant. The authors commented, [op. cit., page 393], that an application of Roth theorem for the approximations of algebraic integers, see (15), to the Ramanujan tau function \( \tau(p^n) \) yields

\[ |\tau(p^n)| \gg p^{11(n-4)/2-\varepsilon}, \]  

(9)

where the implied constant depends on \( p^n \geq 1 \), and \( \varepsilon > 0 \). But the implied constant is not computable. Furthermore, the effective lower bound

\[ 2p^{(k-1)n/2} \frac{\log \log p^n}{(\log p^n)^{1/2}} < |\lambda(p^n)|, \]  

(10)

on a subset of primes of density 1, was proved in [4, Theorem 1].

This note applies an explicit version of Liouville theorem for the approximations of algebraic integers to obtain the following result.

**Theorem 1.1.** Let \( \lambda(m) \neq 0 \) be the \( m \)th coefficient of a modular form \( f(z) = \sum_{m \geq 1} \lambda(m)q^m \) of weight \( k \geq 4 \). If the integer \( m = p^n \) is a prime power, and \( \varepsilon > 0 \), then

\[ |\lambda(p^n)| \geq \frac{1}{8} p^{(k-3)n/2-2k+2-\varepsilon}, \]  

(11)

as \( p^{n/2} \to \infty \).

The parameters \( p^{n/2} > p^5 \), and \( k \geq 4 \) produce a nontrivial lower bound. Specifically, for \( k = 12 \), there is a new explicit lower bound

\[ |\tau(p^n)| \geq \frac{1}{8} p^{9n/2-22-\varepsilon}, \]  

(12)

as \( p^5 \leq p^{n/2} \to \infty \).

**2 Basic Results In Diophantine Approximations**

The concept of measures of irrationality of real numbers is discussed in [11, p. 556], [1, Chapter 11], et alii. This concept can be approached from several points of views.

**Definition 2.1.** The irrationality measure \( \mu(\alpha) \) of a real number \( \alpha \in \mathbb{R} \) is the infimum of the subset of real numbers \( \mu(\alpha) \geq 1 \) for which the Diophantine inequality

\[ |\alpha - \frac{p}{q}| \gg \frac{1}{q^{\mu(\alpha)+\varepsilon}} \]  

(13)

where \( \varepsilon > 0 \) is an arbitrary small number, holds for all large \( q \geq 1 \).
Let $\alpha$ be a root of an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree $\deg f = d$. The Liouville inequality
\[
|\alpha - \frac{p}{q}| > \frac{c}{q^d},
\]
where $c > 0$ is a constant, is the oldest result for irrationality measure of an irrational number as $\alpha$. The Liouville result was superseded by Thue-Roth theorem for the approximations of algebraic integers
\[
|\alpha - \frac{p}{q}| > \frac{c(\alpha, \varepsilon)}{q^{d+\varepsilon}},
\]
where the constant $c(\alpha, \varepsilon) > 0$ is not computable.

A completely explicit version of the Liouville theorem is consider here. The proof given below is relatively recent, an older version appears in [9, Theorem 7.8]. The preliminary definitions are a few concepts used in the proof.

**Definition 2.2.** Let $r = a/b \in \mathbb{Q}^\times$ with $\gcd(a, b) = 1$, be a rational number. The *height* is defined by
\[
H(r) = \max\{|a|, |b|\}.
\]

**Definition 2.3.** Let $\alpha$ be a root of the irreducible polynomial
\[
f(x) = a_d (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d).
\]
The *Mahler height* is defined by
\[
H(\alpha) = |a_d| \prod_{1 \leq i \leq d} \max\{1, |\alpha_i|\}.
\]

**Theorem 2.1.** (Liouville) Let $\alpha$ be a real algebraic number of degree $d \geq 1$. There is a constant $c(\alpha) > 0$ depending only on $\alpha$ such that
\[
|\alpha - \frac{p}{q}| \geq \frac{c(\alpha)}{H(p/q)^d},
\]
where $c(\alpha) \geq 2^{1-d}H(\alpha)^{-1}$, for every rational number $p/q$ if $d \geq 2$.

*Proof.* (Same as [3 Theorem 8.1.1]) Let $f(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of $\alpha$. Take $p/q \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$, and consider the number
\[
F(p, q) = q^d f(p/q) = a_dx^d + a_{d-1}x^{d-1}q + \cdots + a_1xq^{d-1} + a_0q^d.
\]
By assumption, $f(p/q) \neq 0$. Hence, $F(p, q)$ is a nonzero integer. So $|F(p, q)| \geq 1$. To obtain an upper bound for $|F(p, q)|$, rewrite it in factored form
\[
F(p, q) = a_d (p - \alpha_1 q)(p - \alpha_2 q) \cdots (p - \alpha_d q),
\]
and estimate each factor as follows. Let $\alpha = \alpha_1$, then
\[
1. |p - \alpha q| = \left| \frac{p}{q} - \alpha \right| |q| \leq \left| \frac{p}{q} - \alpha \right| H(\alpha),
\]
\[
2. |p - \alpha_1 q| \leq 2 \max \{1, |\alpha_1|\} \cdot \max \{|p|, |q|\} \leq 2 \max \{1, |\alpha_1|\} H(p/q),
\]
where \( i = 2, 3, \ldots, d \). Thus,

\[
|F(p, q)| = |a_d| \prod_{1 \leq i \leq d} |p - \alpha_i q| \leq |a_d| \left| \frac{p}{q} - \alpha \right| \cdot H(p/q)^d \cdot 2^{d-1} \cdot \prod_{2 \leq i \leq d} \max \{1, |\alpha_i|\} \leq |a_d| \left| \frac{p}{q} - \alpha \right| \cdot H(p/q)^d \cdot 2^{d-1} \cdot H(\alpha).
\]

(22)

Combined the lower bound \( |F(p, q)| \geq 1 \) and the inequality (22) to complete the proof. ■

**Corollary 2.1.** Let \( \beta \) be an algebraic number of degree \( d = 2 \) and height \( H(\beta) \leq 4p^{2(k-1)} \). Let \( \{p_m/q_m : m \geq 1\} \) be a subsequence of convergents of height \( H(p_m/q_m) \leq p^{n/2} \), where \( p^n \) is a large prime power. Then,

\[
\left| \beta - \frac{p_m}{q_m} \right| \geq \frac{1}{8p^{n+2k-2}}.
\]

(23)

as \( q_m \leq p^{n/2} \to \infty \).

**Proof.** Let \( \beta \) be a root of the polynomial \( f(T) = a_2 T^2 + a_1 T + a_0 \) of degree \( \deg f = 2 \), and coefficients \( |a_0|, |a_1| \leq p^{k-1} \), \( a_2 = 1 \). These data imply that \( |\beta| \leq 2p^{k-1} \), and the height is at most

\[
H(\beta) = |a_d| \prod_{1 \leq i \leq d} \max \{1, |\beta_i|\} \leq 4p^{2(k-1)},
\]

(24)

see Definition 2.3. Hence, the constant in inequality (19) has at least the value

\[
c(\beta) \geq 2^{1-d} H(\beta)^{-1} = 2^{1-2} \cdot (4p^{2(k-1)})^{-1} = \frac{1}{8p^{2(k-1)}},
\]

(25)

and the height of the rational approximation is

\[
H(p_m/q_m) \leq p^{n/2}.
\]

(26)

An application of Theorem 2.1 yields

\[
\left| \beta - \frac{p_m}{q_m} \right| \geq \frac{c(\beta)}{H(p_m/q_m)^d} \geq \frac{1}{8p^{2(k-1)} (p^{n/2})^2} \geq \frac{1}{8p^{n+2k-2}}.
\]

(27)

**Corollary 2.2.** Let \( \beta = \alpha_{p^{n+1}}^{(n+1)} \) be an algebraic number of degree \( d = 2 \) and height \( H(\beta) \leq 4p^{2(k-1)} \). Let \( \{p_m/q_m : m \geq 1\} \) be a subsequence of convergents of height \( H(p_m/q_m) \leq p^{n/2} \), where \( p^n \) is a large prime power. Then,

\[
|\beta - 1| > \frac{1}{8p^{n+2k-2}}.
\]

(28)

as \( q_m \leq p^{n/2} \to \infty \).
Proof. The inverse triangle inequality, and Corollary 2.1 lead to
\[
|\beta - 1| = \left| \beta - \frac{p_m}{q_m} - \frac{p_m}{q_m} \right| \\
\geq \left| \beta - \frac{p_m}{q_m} \right| - \left| 1 - \frac{p_m}{q_m} \right| \\
\geq \left| \beta - \frac{p_m}{q_m} \right| \\
\geq \frac{1}{8p^{n+2k-2}},
\]
since
\[
\left| 1 - \frac{p_m}{q_m} \right| = \left| \frac{q_m - p_m}{q_m} \right| \geq \frac{1}{p^{n/2}},
\]
as \(q_m \leq p^{n/2} \to \infty\).

3 The Main Result

Proof. (Theorem ) Let \(\alpha_p = p^{(k-1)/2}e^{i\theta_p}\), where \(0 \leq \theta_p \leq \pi\), be the root of the polynomial
\[
f(T) = a_2T + a_1T + a_0 = T^2 - \lambda(p)T + p^{k-1}.
\]
Modify the Binet formula (for integers sequences of the second order defined by a quadratic polynomial) into the following form
\[
\lambda(p^n) = \frac{\alpha_p^{n+1} - \overline{\alpha_p}^{n+1}}{\alpha_p - \overline{\alpha_p}}
\]
\[= \frac{\alpha_p + \overline{\alpha_p}}{\alpha_p + \overline{\alpha_p}} \cdot \frac{\alpha_p^{n+1} - \overline{\alpha_p}^{n+1}}{\alpha_p - \overline{\alpha_p}}
\]
\[= (\alpha_p + \overline{\alpha_p}) \cdot \frac{\alpha_p^{n+1} - \overline{\alpha_p}^{n+1}}{\alpha_p^2 - \overline{\alpha_p}^2}
\]
\[= (\alpha_p + \overline{\alpha_p}) \cdot \frac{\alpha_p^{n+1} - \overline{\alpha_p}^{n+1}}{\alpha_p^2 - \overline{\alpha_p}^2} \left( \alpha_p^{n+1} \overline{\alpha_p}^{-(n+1)} - 1 \right).
\]
Replace the algebraic integer \(\alpha_p = p^{(k-1)/2}e^{i\theta_p}\), where \(0 < \theta_p < \pi\). Taking absolute value and simplifying yield
\[
|\lambda(p^n)| = \left| p^{(k-1)/2} \cos(\theta_p) \cdot \left| \frac{p^{(k-1)(n+1)/2}e^{i\theta_p}}{2p^{(k-1)/2} \sin(2\theta_p)} \right| \right| \alpha_p^{n+1} \overline{\alpha_p}^{-(n+1)} - 1 \right| \\
\geq p^{(k-1)n/2} \left| \alpha_p^{n+1} \overline{\alpha_p}^{-(n+1)} - 1 \right|,
\]
since \(\lambda(p) \neq 0\) implies \(\theta \neq \pi/2\). Replacing the estimate in Corollary 2.2 for \(n \geq 1\), yields
\[
p^{(k-1)n/2} \left| \alpha_p^{n+1} \overline{\alpha_p}^{-(n+1)} - 1 \right| \geq p^{(k-1)n/2} \cdot \frac{1}{8p^{n+2k-2}}
\]
\[\geq \frac{1}{8p^{(k-3)n/2-2k+2-\varepsilon}},
\]
where \(\varepsilon > 0\).
\[\square\]
References


